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On Cubic Surfaces and Their Nodes, ... *S. Lefschetz,*

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On Cubic Surfaces and Their Nodes.

BY S. LEFSCHETZ.

1. The present paper is chiefly concerned with the development of a construction of non-ruled cubic surfaces given by R. Sturm(1) It is first shown rapidly how from this construction the whole theory of the surface can be obtained by applying the principle of correspondence, then the modifications necessary to obtain the 21 different species are given. While very few of the results obtained are new, the general treatment is, as well as the use of synthetic methods to obtain constructions of nodal cubic surfaces. Furthermore, these processes are susceptible of extension to surfaces of higher order, but we reserve this for a later occasion.

2. Before entering upon our main subject we will treat the following problem: *To find the number of common bisecants to two arbitrary twisted curves.*(2) Let R_m, R_n be the two curves, m, n their order, δ_m, δ_n the number of double edges of their projecting cone from an arbitrary point in space. It will be sufficient for our purpose to consider the case where three of the common bisecants are trisecants for either curve. Let then A be a point on R_m, from which we draw one of the bisecants of R_n that go through it, and meet R_n in a' and a''. Through either of the latter points draw one of the bisecants of R_m, which will meet it in 2 points, one of which we will call B. To every point A there correspond $2\delta_n$ points a, and therefore $4\delta_m\delta_n$ points B. Also to every point B there correspond all the points in which R_n meets the cone of order $(m-1)$ projecting R_m, from B, and they are $n(m-1)$in number. Similarly to every

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1. Flachen dritter Ordnung. p. 338.

2. Another solution given in Basset Geometry of Surfaces. p. 230.

point a there correspond $m(n-1)$ points A, hence to B there correspond $m\,n(m-1)\,(n-1)$ points A. It follows that A and B are in a correspondence $[4\delta_m\,\delta_n,\ m\,n(m-1)\,(n-1)]$, and since if A is arbitrary on R_m, one of the points B coincides with it, the number of coincidences is $[4\delta_m\,\delta_n+m\,n(m-1)\,(n-1)]$. It is seen at once that when A coincides with B, $A\ a$ is a common bisecant, and further that if A' is the other end of the latter, then two points B coincide with A, and two of them with A', so that to a common bisecant there corresponds four coincidences in the above correspondence, and therefore the number of common bisecants is

$$\delta_m\,\delta_n+\frac{mn\,(m-1)\,(n-1)}{4}$$

In particular suppose $m=n=3$, then we know that $\delta_m=\delta_n=1$, and therefore

$$\delta_m\,\delta_n+\frac{mn\,(m-1)\,(n-1)}{4}=10.$$

In other words *two twisted cubics have ten common bisecants.*

3. Let now R_1^3 and R_2^3 be two twisted cubics having a common point O. They still have ten common bisecants. For the only thing that could change this circumstance, would be multiple coincidences in O, and that this is not the case can be shown very simply by considering two twisted cubics having five common points, and ten common bisecants, namely the lines joining these points two by two. Now the quadric cones projecting the two curves from O have four generators in common which are common bisecants, and therefore: *When two twisted cubics go through the same point, they have six common bisecants not going through this point.*(3) We will denote these lines by a_1,.....a_6. Let now P be an arbitrary plane through O, OD any line in P, Σ_1, and Σ_2 the two quadrics defined respectively by R_1^3, and OD, R_2^3 and OD. Their intersection is composed of OD and a twisted cubic r^3. When OD describes the plane P, the two quadrics describe two projective pencils of quadrics, and therefore their intersection has for its locus a quartic surface. But P is a part of the latter, hence the locus of r^3 is a cubic surface S_3, which will contain R_2^3, R_1^3 and the lines a_1.....a_6 as having four points in common with each of them. This surface S_3 goes through a fixed curve of order $3+3+6=12$, when P varies, and therefore is absolutely independent of the plane P. Hence *two twisted cubics that meet in a point define one and only one cubic surface.*

3. R. Sturm. Die Lehre von den Geometrischen Verwandschaften. Vol. II, p. 188.

4. We will now proceed to show rapidly how the fundamental properties of the surface can be deduced from the preceding generation. If $i \neq k$ then a_i and a_k have no point in common, since they are bisecants of a twisted cubic. No quadric can contain four of the lines a, for all its generators of the other system would then be on S_3, which would then decompose. Let Σ be the quadric defined by a_1, a_2, a_3. A plane through a_1, is tangent to S_3 in the two points where a_3 is met by its residual conic, and to Σ in one point. The points of contact of such a plane with the two surfaces are therefore in a (1, 2) correspondence, and its 3 coincidences show us, that the residual intersection of Σ and S_3 is formed of three straight lines. On the other hand there are two generators of Σ that meet a_4, and similarly for a_5 and a_6, while such generators are common to S_3 and Σ, as having four points on S. The only way to satisfy these two conditions, and those obtained by a cyclic substitution on the a's, is to have six lines $b_1, \ldots\ldots\ldots b_6$, such that each meets all the lines a, of different index. We have thus obtained the double six of Schlaefli.(4) The plane $a_i b_k$ meets S_3 in a third line c_{ik}, while it meets a_k in a point that must be on c_{ik}, since it is neither on a_i nor b_k. We have thus 15 new lines c_{ik} $(i=1, \ldots\ldots 6,\ k=1, \ldots\ldots 6)$, since c_{ik} coincides with c_{ki}, and therefore the 27 lines of the surface. Since a_i meets b_k it does not meet c_{ik}, which therefore cuts the plane $a_i b_l$ in a point necessarily on c_{il}.

The cubics r^3 do not cut a_i, for the latter meets Σ, in two points on R_1^3, and therefore nowhere else. But b_i cuts Σ_1 in two points which are not on R_1^3, or else, being a bisecant of R_1^3 it could not meet a_k, and therefore these points are on r^3. Finally the plane $(a_i b_k c_{ik})$ cuts R_1^3 in three points on S^3, and since one is on b_k, and only two on a_i, the third must be on c_{ik}, which therefore meets R_1^3, and as well R_2^3. Then R_1^3 cuts Σ_2 in six points one of which is O, therefore it meets r^3 in five points. The curves r^3 form a linear series on S_3, which is ∞^2 like OD. Through any point of the surface there go ∞^1 such curves, and since r^3 is residual of R_1^3 on δ_3 their relation is entirely reciprocal, and shows the existence of two sets of ∞^2 twisted cubics on S^3 corresponding to the double six $(a_i)(b_k)$. The system (R_1^3, R_2^3) depends upon 23 arbitraries, since each one of these curves is determined by six points, and that O is variable on one of them. On the other hand O is arbitrary on S_3, and through it there go ∞^1 curves R^3, hence S_3 depends upon $23-2-2.1=19$ arbitraries.(5) If the absolute invariants are taken as parameters,

4. Quarterly Journal. Vol. II, p. 115.

5. R. Sturm. Die Lehre von den Geom. Verwandschaften. p. 193.

and if we consider only classes with respect to the general projective group, we can say that there are only $\infty^{19-15} = \infty^4$ classes of S_3.

5. ([1])Suppose now that R_1^3 and R_2^3 have, besides O, another common point D. Let DT_1 and DT_2 be their tangents in D. Then Σ_1 and Σ_2 pass both through D, which will be a fixed point of the curve r^3 generating the surface. The tangent planes to the two quadrics in D_1 pass respectively through DT_1 and DT_2, and are projectively related. Therefore their intersection, which is tangent to r^3 in D, describes a quadricone. Hence D is a conic node of the surface. Two twisted cubics cannot have more than five points in common, since only one goes through six points. *Hence a cubic surface has at most four nodes.* The construction of surfaces with 1 , 2. 3, 4 nodes follows at once.

If the surface has only one node say D, three of the lines a go through D, and are obtained by projecting R_1^3 and R_2^3 from D, and taking the three common generatrices of the projecting cones other than OD. Let a_1, a_2, a_3 be these lines, then a_4, a_5, a_6 are the three remaining common bisecants or : *Two twisted cubics having two points in common have three common bisecants not going through either common point.* It follows readily that three of the lines b pass through D, and also that all these lines through D are binary.

Let now the *surface have two nodes, D_1 and D_2.* Then it is seen at once that of the lines a, for example a_1 coincides with $D_1 D_2$, a_2 and a_3 go through D_1, a_4 and a_5 through D_2, hence only a_6 is left as common bisecant of R_1^3 and R_2^3, going through one of their common points. But b_6 must meet a_1.....a_5, hence it coincides with a_1, and b_1 meets a_2.....a_6, hence it is the intersection of the planes (a_2, a_3) and (a_4, a_5) and meets a_6. Hence this proposition: *When two twisted cubics have three common points they have one and only one common bisecant going through none of them. Their 6 common bisecants going through only one of the common points form three planes that meet on the preceding bisecant.* $D_1 D_2$ which is called the *axis* is easily seen to be a *quaternary line*, the 4 other lines through either nodes are binary, so that there are $27-4-4.2.2=7$ *unary* or simple lines.

Nothing particular can be said about surfaces with three or four nodes, except that their properties are readily deduced from our construction, but it is not worth while entering into any details.

6. Suppose now that the plane T_1DT_2 corresponds to itself, in the projectivity between the tangent planes of Σ_1 and Σ_2 in D. Then the tangent cone in D decomposes into the plane T_1DT_2 and another,

6. For the surfaces with nodes, we will follow the usual relation. See Salmon, Geometry of Three Dimensions, 4th ed., p. 489.

and we obtain a binode B_3. This will happen when for a certain position, Σ_1 and Σ_2 are tangent in D. If O_3 is their common generator through O, cutting their common tangent plane T_1DT_2 in Q, then DQ is on both quadrics and therefore on S_3. Furthermore on either Σ_1 or Σ_2, DQ is of the other system than OQ, hence it is a bisecant of R_1^3, and of R_2^3 as well. Conversely if DQ common bisecant of R_1^3 and R_2^3, is in the plane T_1DT_2, and if Q is any point on DQ, then DQ is on both quadrics and therefore on S_3. Furthemore on either Σ_1 or Σ_2, DQ is of the other system than OQ, hence it is a bisecant of R_1^3, and of R_2^3 as well. Conversely if DQ common bisecant of R_1^3 and R_2^3, is in the plane T_1DT_2, and if Q is any point on DQ, then there exists a surface Σ_1 going through OQ, and it will contain DQ, since it meets it in three points. The same will be true of Σ_2, and therefore these two surfaces will be both tangent in D to the plane T_1DT_2 which will correspond to itself in the projectivity of the tangent planes. Hence: *the necessary and sufficient condition for D to be a biplanar node of S_3, is that the plane of the tangents of R_1^3 and R_2^3 in D contain one of their common bisecants through D.*

Therefore if we propose to construct a cubic surface having *a biplanar node* we can give ourselves three coplanar and collinear lines DT_1, DQ, DT_2, two points M_1 and M_2 on DQ, and an arbitrary point O. R_1^3 will then be subjected to the condition of passing through O, D, M_1, and be tangent to DT_1, and R_2^3 to that of passing through O, D, M_2 and be tangent to DT_2. This offers no difficulty whatsoever. In particular if M_1 is made to coincide with M_2 *then we have a surface* (B_3+C_2) that is having a conic node and a binode. By making both R_1^3 and R_2^3 pass through a fourth point quite arbitrary, the construction is still possible and we *have the surface* (B_3+2C_2) having two conic nodes and a binode.

Consider now the points O, D_1, D_2. Through D_1 draw two lines $D_1T'_1$, and $D_1T'_2$ such that their plane contain D_1D_2, then through D_2 two arbitrary lines D_2T_1'', D_2T_2'' and in their plane a line D_2Q on which we will take two arbitrary points M_1 and M_2. We know how to construct a twisted cubic through O, D_1, D_2, M_1 and tangent to D_1T_1'', and D_2T_1''—it is only a special case of the construction of a twisted cubic through 6 points—let it be R_1^3. Similarly we can construct R_2^3 through $OD_1D_2M_2$, and tangent to D_2T_2' and D_2T_2''. *The two curves define a surface* $(2B_3)$ that is with two binodes. If now M_1 and M_2 coincide, which does not affect the construction of the surface, *we have the case* $(2B_3+C_2)$, or a surface with two binodes and one node.

Let us now try to construct a *cubic surface having three binodes* that is the surface $(3B_3)$. Consider a tetrahedron $SABC$, and draw the lines AT_1', AT_2'' in the plane SAB, BT_1'', in the plane SBC, CT_1''' in the plane SCA. We can construct a twisted cubic R_1^3 going through A, B, C and tangent to AT_1', BT_1'', CT_1'''. If we take then a point O arbitrarily on R_1^3 we may remark that R_2^3 is perfectly determined by the conditions that it pass through O, A, B, C, be tangent to AT_2' and to the planes SBC and SCA in B and C. For let E be the point where AT_2' meets SB. Then the conic projection of R_2^3 on the plane SBC must go through ω, A B, C, E and be tangent to SC, ω being the projection of O on the plane SBC, while the conic projection of R_2^3 on the plane SAB must go through A, B, $\omega'$and be tangent to SB and AE. R_2^3 is thus perfectly determined as the intersection of the projecting cones from two of its points.

7. Suppose now that R_1^3 and R_2^3 be tangent in D, and let DT be their common tangent. Then the two projective pencils of the tangent planes of Σ_1 and Σ_2, have DT for common axis. The double planes of the projectivity in question will be the tangent planes to S_3 in D, and their intersection DT will lie on the surface. *D will therefore be a binode B_4* that is such that the intersection of the biplanes be on the surface. This is all clearly shown when the contact of the two curves is considered as the limit of two common points infinitely near each other, and this consideration shows also that the point B_4 is equivalent—as far as the class of the surface goes—to two ordinary nodes. There is no difficulty therefore in constructing the surface B_4, having a binode in D. We give ourselves DT and O, and have to construct two twisted cubics going through O, D and tangent to DT. By subjecting the two curves to the condition of having one or two more common points, which can be chosen arbitrarily we obtain the surfaces $(C_2 + B_4)$ and $(2C_2 + B_4)$.

We can easily obtain the *lines on the surface B_4* by direct considerations(7). There are two common bisecants of R_1^3 and R_2^3 through D, each of them counting for two, since there is contact of the two curves in D, while DT is also on S_3. We thus have a_1,.....a_5, therefore R_1^3 and R_2^3 have a bisecant a_6 not going through D. Suppose that $a_2 \equiv a_3$, $a_4 \equiv a_5$. Then the two biplanes are (DT, a_2) and (DT, a_4), and if they are met by a_6 in E and F, the lines DE and DF are also on S_3. Let now P be an arbitrary plane through DT, it meets R_1^3, R_2^3 and a_6 in points G_1, G_2 and H, and HG_2 meet respectively DT' in points K_1,K_2, which are seen at once to be in

7. Cailey, A Memoir on Cubic Surfaces. Phil. Transacts. 1869. p. 231. Schlaefli, Ibid. 1863. p. 193.

(1, 1) correspondence, and the two coincidences of the latter show that there are two lines meeting R_1^3, R_2^3, DT and a_6. Of these one goes through O, and the other a *transversal* t meets the surface S_3 in four points and therefore belongs to it. The plane (DE, a_4) meets R_1^3 and R_2^3 in the two points not on a_2 and defining a line e on S_3. Similarly the plane (DF, a_2) defines a line f on S_3. both these lines meeting t in the points where it cuts the planes (DE, a_4) (DF, a_2) respectively. Finally the plane (a_2, a_4) meets a_6 and t in two points defining a third line g. This system of lines is identical with that found by Schlaefli as we should expect.

8. Consider now the three points O, D_1, D_2, and three coplanar lines D_2T_1, D_2T_2 and $D_2M_1M_2$. We could define an S_3 having a C_2 in D_1 and a B_3 in D_2 by considering the twisted cubics R_1^3 and R_2^3, the first passing through $OD_1D_2M_1$ and tangent to D_2T_1, and the second O, D_2, $D_{,2}M_2$ and tangent to D_2T_2. We wish to find *what will happen when D_2 is made to approach D_1 indefinitely.* If we consider the two conics Γ_1 and Γ_2 going through D_1 and D_2 and tangent to R_1^3 and R_2^3 respectively, in D_2, there are ∞^1 quadrics going through them, all tangent to the plane $T_1D_2T_2$ in D_2. At the limit Γ_1 and Γ_2 will osculate R_1^3 and R_2^3, and the plane tangent to the quadrics containing both of them will contain a common bisecant of R_1^3 and R_2^3 through D. We can also say that the tangent plane in D_1 to any quadric osculating both R_1^3 and R_2^3 at that point contains a common bisecant. *Hence the following construction.* Let DT be a tangent to a given quadric Σ, through DT consider two planes cutting Σ in the conics Γ_1 and Γ_2, and in the plane P tangent to Σ in D take two points M_1, M_2 collinear with D. If O is an arbitrary point in space, R_1^3 will go through O, D, M_1, and osculate Γ_1 in D, while R_2^3 will go through O, D, M_2 and osculate Γ_2 in D. The construction presents no difficulty—R_1^3 for example is determined by its two projecting cones from D and O, the first being tangent to the plane of Γ_1 along DT and going through M_1 and O, while the second osculates Γ_1 in D and goes also through M_1. *We thus have the surface with a binode B_5* equivalent to $(C_2 + B_3)$.

In particular suppose that the planes of Γ_1 and Γ_2 coincide, or that R_1^3 and R_2^3 have the same osculating plane, in D. Then M_1 and M_2 will be the points infinitely near D on the respective cubics, and can still be considered as being collinear with D. It is clear that the preceding case is obtained as the limit of the one in which D_1 is a binode B_3 as well as D_2, for then D_1D_2 is in the plane of the tangents to the curves in D_1, and the latter becomes

therefore a common osculating plane at the limit. *This construction will then give the binode* B_6 equivalent to $2B_3$.

If in the two preceding constructions the two cubics were subjected to pass through another fixed point D_3 we would obtain *the surfaces* $(C_2 + B_5)$ *and* $(C_2 + B_6)$.

The foregoing conclusions are easily verified by analysis. Let us take as an example the case of B_5. This surface has for equation[8]:

$$WXZ + Y^2Z + YX^2 - X^3 = 0 \qquad (1)$$

If we consider the following ∞^2 system of quadrics:

$$XY + (lX + mZ)(Y + Z) = 0 \qquad (2)$$

$$(X - W)(lX + mZ) - (Y - Z)Z - XW = 0 \qquad (3)$$

it represents ∞^2 twisted cubics on (1), for if we eliminate $(lX+mZ)$ between (2) and (3) we obtain (1), while (2) and (3) have in common $X=Z=0$. (2) represents the projection of R^3 general cubic of the system on $W = 0$, and as R^3 is tangent to $X=Z=0$ at the point (0, 0, 0, 1), the osculating plane at the latter has for equation $X + (lX + mZ) = 0$. From (3) we deduce that it meets R^3 on the quadric. $X^2 + Z(Y-Z) = 0$ which is a cone tangent to $Z=0$, in which $Y=Z=0$ fixed bisecant of R^3 through (0, 0, 0, 1) or D is situated. Any two curves R_1^3 and R_2^3 of the system considered satisfy therefore the condition that they are tangent in a point D, and that one of their common bisecants through D touches a quadric osculating them at that same point, as we found above.

We may remark that the plane P touching in D the quadric osculating R_1^3 and R_2^3 is evidently one of the biplanes.

9. If in the construction for S_3 with a binode B_4, the cones projecting R_1^3 and R_2^3 from the latter are tangent we obtain a binode in which the biplanes coincide that is a *U-node* U_6. For if D is the binode, DT the tangent to both curves, the biplanes are as we have seen, the planes going through DT and the two common bisecants of R_1^3 and R_2^3 through D, different from OD, and they both coincide in the case considered with a_2. R_1^3 and R_2^3 still have a bisecant a_6 going through neither D nor O, and there is still a transversal t as for B_4. If a_6 meets the U-plane in E, DE is a line on S_3, and finally the plane (a_6, t) meets the plane tangent to the projecting cones of R_1^3 and R_2^3 along a_2 in a line g on the surface. This last result is obtained by considering the limiting position of the plane (a_2, a_4) when a_4 approaches indefinitely a_2. We thus have

8. Basset Ibid. p. 61.

obtained for U_6 directly the well known system of lines existing on the S_3 having such a point. But it will be well to verify our conclusions by analysis, for we have shown here that U_6 on a cubic surface is not equivalent to $3C_2$, but is a special singularity.

If

$$W(X+Y+Z)^2 + XYZ = 0 \qquad (1)$$

is the equation of an S_3 having an U_6 in (0, 0, 0, 1), then it is seen at once that:

$$X[h(X+Y+Z) - kY] - (X+Y+Z)^2 = 0 \qquad (2)$$

$$W[h(X+Y+Z) - kY] + YZ = 0 \qquad (3)$$

represent ∞^2 twisted cubics R^3 on S_3. The cone projecting them from the U-node are prepresented by (2) and they are tangent to $X=0$, where it is met by $X+Y+Z=0$. These R^3 meet $X+Y+Z=0$ and $Z=kW$ the first being a point of contact as can easily be seen. The line $X=W=0$ is effectively on the surface, and it is the intersection of the fixed tangent plane $X=0$ to the cones (2), with the plane $W=0$ which is the plane containing $W=Y=0$, and $W=Z=0$, the only other two lines of the surface, not going through the node. Our conclusions obtained by direct considerations are thus verified.

10. If in the preceding construction for U_6, R_1^3 and R_2^3 have the same osculating plane in D, and still the same tangent D Tat that point, then a_2 will coincide with DT and *we will have obtained a surface with a node* U_7, such that the U-plane contains only two lines, while the only other line on the surface will be a_6, which meets the common osculating plane in a point E, situated on DE, the second line of the U-plane. We will not carry out the analytical verification, which is as easy as before.

The surface with a node U_8 is the only one which cannot be obtained by the method used so far. The reason of it, *is that this surface*, as we shall prove now, *contains no twisted cubics*. For as is well known, there is only one straight line D on such a surface, and it is obtained by the coincidence of the 27 lines of the ordinary one. Hence a twisted cubic R^3 on our surface must meet D in two points. Now there are ∞^1 quadrics going through R^3 and D, and they must meet S_3 in ∞^1 conics. Anyone of the latter has only D for residual intersection, and is therefore necessarily in a plane going through D, the latter meeting the corresponding quadric in a line of the third order, is part of it. Hence the ∞^1 quadrics considered are necessarily decomposed into two planes, which shows that R^3 on a cubic surface

with U_8 is necessarily a plane curve. We will give below a construction for this surface.

11. It has been shown by Steiner([9]) that when a pencil of planes and a pencil of quadrics are projectively related, the intersection of the corresponding surfaces generates a cubic surface. It can be proved as in §5 that when the bases of the two pencils have a common point, the latter is a node of the S_3. We will not treat all the possible cases this way, as we doubt if the generality that we have reached can thus be obtained easily. Let us consider only the case where the axis of the quadrics is composed of two conics, having then necessarily two common points D and D', while the axis DE of the pencil of planes goes through D, and is in T fixed tangent plane of the pencil of quadrics in D. If Q_1 and Q_2 are the planes of the two conics, they form together one of the quadrics of the pencil, to which will correspond a plane P in the pencil of planes. The lines (P, Q_1) and (P, Q_2) are on S_3.

Further if Σ is the quadric corresponding to the plane P in the plane-pencil, then the two lines l_1 and l_2 of this quadric in P are also on S_3. It follows that since DE is also on S_3, the latter has in D a binode, since the lines on the surface through D, are in two planes through DE, which is then the edge of the binode. The latter is clearly a binode B_4, since its edge DE is on S_3.

Suppose now that T corresponds in the plane-pencil to the system (Q_1, Q_2) in the pencil of quadrics. *There $T \equiv P$ and the surface has in D a point U_6.* The three lines on S_3 through D are DE, (T, Q_1) and (T, Q_2). Hence if in the same conditions DE coincides with (T, Q_1) that is, is tangent to one of the conics in D, *we have the surface with a node U_7*, and if finally the two conics be both tangent to DE in D, then *we have a surface with a node U_8*, the U-plane of which will be the plane tangent to the quadrics osculating both conics in D, or, if we prefer the plane tangent in D to a cone going through both given conics. We thus have a gometric construction for the surface U_8.

Analytically the general equation of an S_3 with a node is

$$WX^2 + X\phi_2(y.z) + \phi_3(Y^1Z) = 0$$

which can be generated by the system:

$$WX + \phi_2 - \lambda\phi'_2 = 0 \qquad (1)$$

$$\lambda X = aY + bZ, \text{ if } \phi_3 \equiv (Y + bZ)\phi'_2, \qquad (2)$$

which shows that the axis of the plane-pencil $X=O$, $aY+bZ=0$ is in $X=0$, fixed tangent *plane of the pencil of quadrics, and goes through its contact* (0, 0, 0, 1).

9. R. Sturm. Flachen Dritter Ordnung. p. 9.